Transition curves for the variance of the nearest neighbor spacing distribution for Poisson to Gaussian orthogonal and unitary ensemble transitions

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Using an appropriate 2×2 random matrix ensemble, transition curves for the variance of the nearest neighbor spacing distribution are constructed for the Poisson to Gaussian orthogonal and unitary ensemble transitions in terms of an easily identifiable transition parameter. [S1063-651X(99)10209-5]

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The classical [Gaussian orthogonal (GOE), unitary (GUE), and sympletic (GSE)] random matrix ensembles are classified by Dyson, and Wigner's surmise gives the nearest neighbor spacing distribution (NNSD) $P(S)dS$ for these ensembles with GOE exhibiting linear and GUE quadratic level repulsion. In addition, these ensembles exhibit spectral rigidity as given by the Dyson-Mehta Δ_3 spectral statistic (see [1,2] and references therein). The other extreme to the GOE, GUE, and GSE spectra are the picket fence (or uniform) and the Poisson spectra. The seminal paper by Bohigas *et al.* in 1984 [3] on the analysis of level fluctuations of the quantum Sinai billiard, whose classical counterpart is known to be completely chaotic, has established that the fluctuation properties of classical random matrix ensembles are generic and therefore applicable for local spectral statistics of all quantal systems (earliest numerical study of this type is due to McDonald and Kaufman [4]). In fact, as Berry states [5], ''If the system is classically integrable Δ_3 corresponds to that of Poisson systems, if the system is classically chaotic and has no symmetry Δ_3 corresponds to that of GUE and if the system is chaotic and has time reversal symmetry Δ_3 corresponds to that of GOE.'' With these, the subject of quantum chaos is developed; see the reviews [6,7].

The changes in the nature of level fluctuations as a symmetry is gradually broken, as two good symmetry subspaces are gradually admixed, as ordered (integrable) spectra gradually become chaotic, etc., are studied by using interpolating and/or partitioned random matrix ensembles [8–11]. In all these situations, one can identify that the transition parameter (Λ) and the measures for level fluctuations such as the variance of NNSD [$\sigma^2(0)$ in the notation of [2]], number variance $\Sigma^2(r)$, Δ_3 statistic, etc., versus Λ give the transition curves. The transition curves for many different types of random matrix interpolations are considered in the literature; see [8–11]. For example, the GOE-GUE transition curve given in [8] is used to derive an upper bound on the amount of time-reversal noninvariant part of nucleon-nucleon interaction [12]. More recently, the transition curve for 2×2 partitioned GOE given in [8,9] is tested by experiments with superconducting microwave billiards [13].

Poisson to GOE and GUE transitions received the attention of a large number of research groups as they represent order to chaos transitions. There are several different formulas, given by Brody [14], Berry and Robnik [15], Hasegawa *et al.* [16], Izrailev [17], Abul-Magd [18], etc., for the NNSD $P_{\text{P-GOE}}(S)dS$ interpolating Poisson and GOE and similarly for $P_{\text{P-GUE}}(S)dS$ interpolating Poisson and GUE [17,19,20]. For example, the Brody distribution [14] $aS^{\beta}\exp\times\{-bS^{\beta+1}\}$, a, and b given easily in terms of the Brody parameter β, is a simple interpolation of Poisson and GOE NNSD's but fits data embarrassingly well. The one (ρ) parameter Berry-Robnik (BR) formulas [15,19] for Poisson to GOE and GUE are applicable when there is only one dominant chaotic region coexisting with regular regions of a dynamical system; ρ is the fractional volume, in phase space, of the chaotic region and $1-\rho$ is the fractional volume of all regular regions put together. Similarly, Hasegawa *et al.* [16] derived their formula for Poisson to GOE by applying a stochastic differential equation approach to the level motion theory, the Izrailev [17] distributions for Poisson to GOE and GUE are based on a generalization of the circular ensemble joint probability distribution for the eigenvalues, etc. However, a simple yet useful approach for deriving the NNSD's is to extend, as pointed out in [21–24], Wigner's 2×2 matrix formalism. Using an appropriate 2×2 random matrix ensemble, transition curves are constructed, for the variance of the NNSD for Poisson to GOE and GUE, in terms of a transition parameter Λ (Λ is mean squared admixing GOE or GUE matrix element divided by β times the square of the mean spacing D_0 of the Poisson spectrum, $\beta=1$ for GOE, and $\beta=2$ for GUE) and these results are reported in this Brief Report. Relationship of the present work to previous studies using 2×2 matrices is pointed out.

Let us consider the following 2×2 matrix [24]:

$$\begin{bmatrix} \alpha(X_1+X_2)+p\nu\lambda & \alpha X_3+i\alpha' X_4 \\ \alpha X_3-i\alpha' X_4 & \alpha(X_1-X_2)-p\nu\lambda \end{bmatrix}. \tag{1}$$

In (1), X_1, X_2, X_3, and X_4 are zero-centered independent Gaussian variables with variance v^2 denoted by $G(0,v^2)$ and the usefulness of p and λ will later become clear. The matrix (1) for $\lambda=0,\alpha'=0$ is GOE, $\lambda=0,\alpha'=\alpha$ is GUE, and $X_i=0$ and λ a Poisson gives a Poisson spectrum. Thus (1) interpolates Poisson, GOE, and GUE. The nearest neighbor spacing distribution for (1) is given by

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$$P(S{:}\lambda)dS=\frac{SdS}{4v^3\alpha^2\alpha'\sqrt{2\pi}}\exp\left(-\frac{p^2\lambda^2}{2\alpha^2}-\frac{S^2}{8v^2\alpha^2}\right)\times\int_0^S dz I_0\left(\frac{p\lambda}{2v\alpha^2}\sqrt{S^2-z^2}\right)\times\exp\left[\frac{(\alpha')^2-\alpha^2}{8v^2\alpha^2(\alpha')^2}z^2\right]. \tag{2}$$

In Eq. (2) I_0 is Bessel function. Equation (2) gives back the GOE-GUE interpolation result derived in [21] (see also [25]) and it is a generalization of Eq. (16) of [21]. Assuming a distribution $f(\lambda)d\lambda$ for λ, Eq. (2) defines, for example, the Poisson to GOE and GUE interpolations. Combining Eq. (2) with

$$P(S)dS=\left[\int_{-\infty}^{+\infty}P(S{:}\lambda)f(\lambda)d\lambda\right]dS,$$

for Poisson,

$$f(\lambda)d\lambda=e^{-\lambda}d\lambda \text{ for } 0\leqslant\lambda\leqslant\infty \quad\text{and}\quad 0 \quad \text{for}\quad \lambda<0 \tag{3}$$

give the spacing distributions for Poisson to GOE and GUE. In fact, with $\alpha'=0$, $p=\sqrt{1/2}$, and $v=\sqrt{1/2}$, Eqs. (2) and (3) give back the Poisson to GOE interpolation formula of [22].

In terms of the transition parameter Λ,

$$\Lambda=\frac{\alpha^2v^2}{D_0^2}, \tag{4}$$

where the mean spacing D_0 of the unperturbed Poisson spectrum is $D_0=2pv$ and the mean square admixing GOE matrix element is α^2v^2, the NNSD is

$$P_{\text{P-GOE}}(\hat{S})d\hat{S}=d\hat{S}\frac{\hat{S}}{4\Lambda}\exp\{-\hat{S}^2/8\Lambda\}\times\int_0^\infty\exp\left\{-\lambda-\frac{\lambda^2}{8\Lambda}\right\}I_0\left(\frac{\lambda\hat{S}}{4\Lambda}\right)d\lambda;$$

$$\hat{S}=S/D_0. \tag{5}$$

For $\Lambda=0$, Eq. (5) gives the Poisson and for $\Lambda\to\infty$ the Wigner (GOE) form. Equation (5) is also derived by Haake *et al.* [22]. It is easily proved that as $\Lambda\to0$ and $\hat{S}$ small, Eq. (5) reduces to

$$P_{\text{P-GOE}}(\hat{S})d\hat{S}=d\hat{S}\sqrt{\frac{\pi}{8}}\frac{\hat{S}}{\Lambda^{1/2}}\exp\left\{-\frac{\hat{S}^2}{16\Lambda}\right\}I_0\left(\frac{\hat{S}^2}{16\Lambda}\right). \tag{6}$$

The formula (6) is derived using a special 2×2 matrix by Caurier *et al.* [23], while Tomsovic [26] and Leyvraz and Seligman [27] derived the same using perturbation theory for a general $N\times N$ matrix. One important result that follows from Eq. (6) is that $P(S)$ goes to zero as S goes to zero for nonzero values of Λ (i.e., there is level repulsion as soon as GOE is switched on). More significantly, the variance of the NNSD,

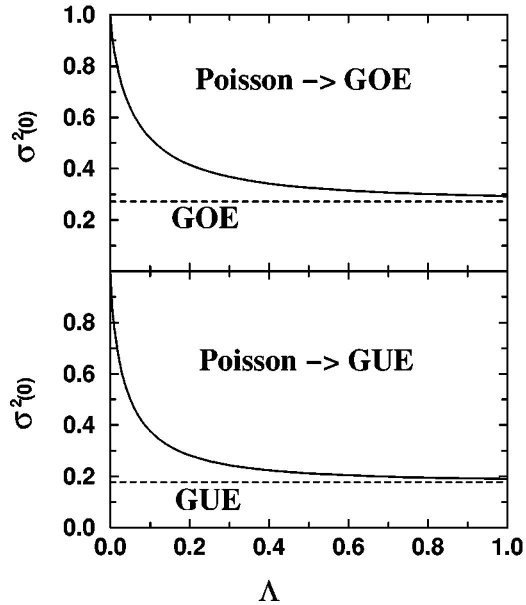

FIG. 1. Transition curves $\sigma^2(0{:}\Lambda)$ vs Λ for Poisson to GOE and Poisson to GUE as given by Eqs. (8) and (12), respectively.

$$\sigma^2(0{:}\Lambda)=(\overline{S^2}/\overline{S}^2)-1, \tag{7}$$

for the Poisson to GOE transition, which defines a transition curve, is given by

$$\sigma^2_{\text{P-GOE}}(0{:}\Lambda)=\frac{8\Lambda+2}{\pi[\Psi(-1/2,0,2\Lambda)]^2}-1. \tag{8}$$

In Eq. (8), Ψ is Kummer's function [28]. The complete transition curve $\sigma^2(0{:}\Lambda)$ vs Λ is given in Fig. 1. It is instructive to consider the small Λ expansion,

$$\sigma^2_{\text{P-GOE}}(0{:}\Lambda)\xrightarrow{\Lambda\ll1}1+4\Lambda[\ln(\Lambda)+1+\gamma-\ln2], \tag{9}$$

where γ is Euler's constant. Note that the $\Lambda\ln\Lambda$ term also appears in the small Λ expansion of the number variance $\Sigma^2(1)$ [8],

$$\Sigma^2_{\text{P-GOE}}(1{:}\Lambda)\xrightarrow{\Lambda\ll1}1+2\Lambda[\ln(\Lambda)+1-\gamma-\ln2]. \tag{10}$$

It is important to mention that $\sigma^2(0)$ and $\Sigma^2(1)$ in (9) and (10) are not simply related as Poisson is involved; see [2]. Finally, the accuracy of (9) is well tested in Fig. 2.

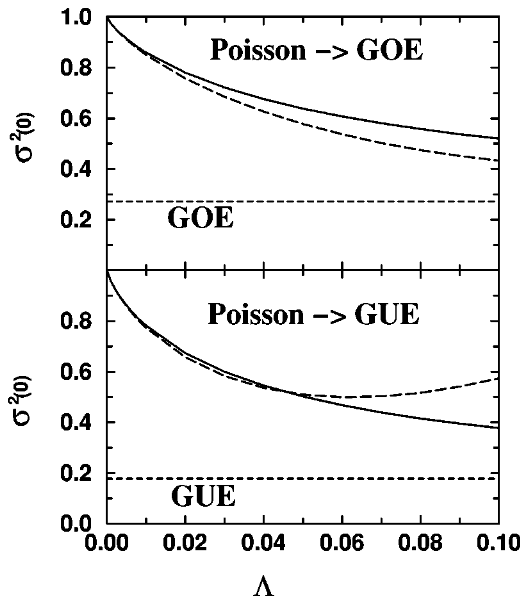

FIG. 2. Transition curves $\sigma^2(0:\Lambda)$ vs Λ for Poisson to GOE and Poisson to GUE for small Λ. Exact results (8) and (12) are compared with the perturbation theory results (dashed curves) given by Eqs. (9) and (13), respectively.

Let us now consider the Poisson to GUE transition. The NNSD for Poisson to GUE is given by [with $\alpha=\alpha'$ in (1) and combining Eqs. (2) and (3)],

$$P_{\text{P-GUE}}(\hat{S})d\hat{S}=d\hat{S}\frac{\hat{S}}{\sqrt{2\pi}\Lambda^{1/2}}\exp\{-\hat{S}^2/8\Lambda\} \times\int_0^\infty \lambda^{-1}\exp\left\{-\lambda-\frac{\lambda^2}{8\Lambda}\right\}\sinh\left(\frac{\lambda\hat{S}}{4\Lambda}\right)d\lambda. \tag{11}$$

It should be noted that the mean squared GUE admixing matrix element is $2\alpha^2v^2$ and hence in this case the transition parameter Λ given by Eq. (4) is the mean squared admixing GUE matrix element divided by two times the square of the mean spacing of the Poisson spectrum. With the NNSD (11), the exact expression for $\sigma^2(0:\Lambda)$ is

$$\sigma^2_{\text{P-GUE}}(0:\Lambda)=\frac{12\Lambda+2}{[X(\Lambda)]^2}-1, \tag{12}$$

$$X(\Lambda)=2\Lambda\left[-\text{Ei}(2\Lambda)+4\sqrt{2\Lambda/\pi}\;{}_2F_2\left(\begin{matrix}1/2,1\\3/2,3/2\end{matrix};2\Lambda\right)\right] +\sqrt{8\Lambda/\pi}+e^{2\Lambda}[1-\Phi(\sqrt{2\Lambda})].$$

In Eq. (12), Ei is an exponential integral and Φ is an error function [28]. Similarly, ${}_2F_2$ is a generalized hypergeometric function. The complete Poisson to GUE transition curve for $\sigma^2(0:\Lambda)$ versus Λ is given in Fig. 1. Once again it is instructive to consider the small Λ expansion,

$$\sigma^2_{\text{P-GUE}}(0:\Lambda)\xrightarrow{\Lambda\ll 1}1+8\Lambda[\ln(\Lambda)+\tfrac{1}{2}+\gamma+\ln 2]. \tag{13}$$

Just as in the case of Poisson to GOE, here also there is the $\Lambda\ln\Lambda$ term. The accuracy of Eq. (13) is tested in Fig. 2; the approximation (13) is good for $\Lambda\lesssim 0.05$.

The transition curves given in Fig. 1 show that the Poisson to GOE and Poisson to GUE transitions are nearly complete for $\Lambda\sim 0.3$. The results in Eqs. (8) and (12) are in fact applicable to general $N\times N$ matrices (or for any interacting many-particle system) through the transition parameter Λ by giving appropriate interpretations to α^2v^2 and D_0 in Eq. (4); this is indeed verified by the results in Fig. 4 of the second paper in Ref. [22]. With this, the results in Eqs. (8) and (12) can be applied to realistic systems. For example, using a sufficient number of energy levels near ground states or near the yrast line at high spins as the case may be in atomic nuclei (similarly in other interacting many-particle systems such as atoms, molecules, etc.), it is possible to deduce the corresponding $\sigma^2(0)$ values. Then from Fig. 1 one can read off the value of Λ (or, depending on the sample size errors, determine a bound on Λ) for Poisson to GOE transitions in these systems, and similarly, the value of Λ for Poisson to GUE transitions in systems without an antiunitary symmetry (see [19] for examples of such systems). It is expected that in some limit the transition parameter Λ should be related to the BR parameter ρ (ρ representing the fractional volume, in phase space, of the chaotic region of a complex dynamical system). Using Fig. 1 and the BR formula for $\sigma^2_{\text{P-GOE}}(0:\rho)$ [Eq. (30) of [15]], it is seen that $\Lambda\simeq\rho/20(1-\rho)$ for $\Lambda\gtrsim 0.05$. However, for $\Lambda\lesssim 0.01$, results of Eq. (8) and the corresponding BR formula differ significantly. Finally, experiments with superconducting microwave billiards similar to those reported in [29] should be able to test the transition curves given in Figs. 1 and 2 just as in [13], where the 2-GOE's to 1-GOE transition is tested.

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$$\sigma^2_{\text{GOE-GUE}}(0:\Lambda)=\frac{\pi(1+3\pi\Lambda)}{[(1+2\pi\Lambda)\tan^{-1}\{(2\pi\Lambda)^{-1/2}\}+\sqrt{2\pi\Lambda}]^2}-1\xrightarrow{\Lambda\ll 1}\left(\frac{4}{\pi}-1\right)-4\Lambda.$$

This formula follows directly from Eqs. (17) and (18) of [21]. Note that $\Lambda=0$ corresponds to GOE and $\Lambda=\infty$ gives GUE.

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